# Portfolio Choice with Time Horizon Risk

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**Abstract**

I study the allocation problem of investors who hold their portfolio until reaching a target wealth. The strategy suppresses final wealth uncertainty but creates a time horizon risk. I begin with a classical mean variance model transposed in the duration domain, then study a dynamic portfolio choice problem with Generalized Expected Discounted Utility preferences. Using long-term US return data, I show in the mean variance model that a large amount of time horizon risk can be diversified away by investing a significant share of equities. In the dynamic model, more impatient investors are also more averse to timing risk and invest less in equities. The optimal equity share is downward trending as accumulated wealth approaches its target.

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# Introduction

Many households invest in financial markets with a personal goal in mind, like saving for retirement, buying a home or financially helping their children. Financial goals can be expressed in two ways, either by setting an investment horizon and obtaining an uncertain amount, depending on market performance, or setting an explicit wealth target, which takes an uncertain time to be attained. The latter uncertainty may suit investors who prefer waiting to missing their wealth target. For example, people undergoing financial losses may choose to defer the purchase of a new car or postpone their retirement date. When feasible, setting a flexible time horizon can be an effective risk management policy. By doing so, investors escape final wealth risk but face in exchange a time horizon (or duration) risk.

A target wealth combined with a flexible time horizon have fundamental consequences for the way investors trade risk for return in financial markets. By investing in equities which return is both higher in expectation and riskier than fixed income assets' return, investors may expect a shorter investment time interval but also a more uncertain delay. Also, the concept of risk aversion has a very different meaning than the usual one when consumption is uncertain. Impatience is a key driver of timing risk aversion as investors have to wait to recoup financial losses.

In this article I study a two-asset portfolio choice problem of an investor who sets a money goal and exits the market when it is attained. Investors are impatient and prefer a shorter time horizon all else equal. In addition, experimental evidence shows that investors are also timing risk averse (Onay and Öncüler, 2007, Dejarnette et al., 2020), i.e. prefer to meet their target wealth in a sure date than in a random one with equal expected delay. Moreover, when faced with two non-degenerate mean preserving date distributions, they prefer the less risky one.

I find that the classical risk-return trade-off faced by investors in the wealth domain does not transpose equally in the duration domain. Using annual return data on US equity and short-term bonds (commercial papers and certificates of deposits) between 1871 and 2019, I present five empirical findings about the comparative benefits of money market assets and equities in terms of expected duration and duration risk. Since bills generate less return on average, they entail longer durations than equities. More surprisingly, they are also riskier than equities in terms of duration variance and skewness. I then present a CAPM-like portfolio choice model in which investors have mean variance preferences over uncertain investment durations. Based on long-period return statistics, I show that a large amount of duration risk can be diversified away by buying a large share of equities.

Next, I study a dynamic and microfounded model of portfolio choice with uncertain time horizon. Investors compose their portfolio with two assets. The first asset is more profitable in expectation and riskier than the second one. The trade-off between expected duration and duration risk is modeled by assuming that investors maximize Generalized Expected Discounted Utility (Dejarnette et al., 2020). In accordance with intuition, impatience plays a key role in the portfolio choice. On the one hand, impatient investors find short-term bond unattractive as low yield means longer expected investment periods. On the other hand, impatience, not marginal utility of wealth across states, is the main driver of risk aversion in the duration domain. I find that more impatient investors are also more timing risk averse. Moreover, the second factor dominates the first one with the result that more impatient individuals invest less in equities. I also show that investors optimally rebalance their portfolio by decreasing the equity share when wealth approaches its target.

Portfolio choice and the optimal combination of risky and safe assets have been extensively studied in the literature, starting with the seminal Capital Asset Pricing Model (CAPM) by Markowitz (1952). Contrary to the CAPM which assumes a fixed time frame, the present paper studies the twin problem of a fixed terminal wealth and an uncertain time horizon. A few articles have investigated related issues. Martellini and Urošević (2006) analyze a Markowitz problem in which investors face an uncertain exit date. Liu and Loewenstein (2002) study an intertemporal portfolio optimization problem with an exponentially distributed time-horizon. Karatzas and Wang (2001) solve the optimal dynamic investment problem when the uncertain time horizon is a stopping time of asset price filtration. Blanchet-Scalliet et al. (2008) extend the setup to a case with a stochastically time-varying probability of exiting the market. Huang et al. (2008) adopt a worst-case conditional value-at-risk approach to manage the exit date. Those articles minimize a measure of downside risk or assume mean-variance or constant relative risk aversion preferences, whereas this article stresses the importance of assuming more general preferences in presence of timing risk. Also, in these articles, exogenous exit dates or exit strategies depend on asset price behavior. Formulating a goal as a target wealth to be met, is simpler and more understandable from the viewpoint of retail investors and is therefore more practitioner-oriented.

The model contributes to the vast literature on dynamic portfolio choice models with a long-term focus (see Campbell and Viceira, 2002, for a survey). Studies generally find that young investors should take more risk than older investors (e.g. Bodie, Merton and Samuelson, 1991, Viceira, 2001, Cocco, Gomes and Maenhout, 2005). When wealth risk is replaced by time horizon risk, investors should not condition the equity share on remaining investment horizon but on wealth which remains to be accumulated to meet their target. Long-term investors set a high money target relative to their current wealth. In both investment frames, investors should invest more in equities the more remote their date or wealth target. However, contrary to popular financial advice with a fixed selling date, the decrease is not linear. The optimal equity share can stay flat for large wealth intervals.

The literature on attitude towards timing risk is sparse and most applications to real world problems are still to be explored. Chesson and Viscusi (2003) note that the Expected Discounting Utility model leads to a counter-intuitive preference for timing risk. Several extensions accounting for timing risk aversion have been proposed, like probability weighting (Onay and Öncüler, 2007), Epstein-Zin (1989) preferences (Dillenberger, Gottlieb and Ortoleva, 2019) or Generalized Expected Discounted Utility (GEDU), recently investigated in depth by Dejarnette et al. (2020). The portfolio choice model assumes GEDU preferences, which are more intuitive and tractable than Epstein-Zin preferences. The initial aim of these preferences was to distinguish the coefficient of relative risk aversion from the elasticity of intertemporal substitution in consumption. The latter is less relevant in problems in which the consumption date, not the consumption level, is uncertain. The GEDU model is also more in accordance with experimental results than the probability weighting model (Dejarnette et al., 2020).

The remainder of the paper is organized as follows. Section 1 presents basic findings about mean-variance trade-offs in the duration domain using long-term US asset returns. Section 2 proposes a simplistic theory of portfolio choice by replacing expectation and risk of return in the Markowitz model by expectation and risk of duration. Section 3 goes beyond the static model and lays out a dynamic portfolio choice model with GEDU preferences. Section 4 simulates the model and interprets the results. Section 5 concludes.

# 1 Empirical findings on investment durations

I compare in this section statistical properties in the duration domain of two main security classes covering the period 1871-2019: US equities and money market securities. For equities, I use data from Cowles (1939) for the period 1871-1925, which include between 12 (1871) and 258 (1925) value weighted securities listed on the New York Stock Exchange. I use the S&P 90 index before 1957 and S&P 500 index afterwards. All returns are annualized, deflated by US CPI inflation rate, and include dividends.

For the market interest rate, I use short term bonds' annualized rates of Shiller (1989, 2015).[1] It consists in the 6-month commercial paper rate published by the Federal Reserve Board until 1997 and the 6-month certificate of deposit rate from 1997 to 2012. The series are completed until early 2020 by data from macrotrends.net and deflated by US CPI inflation rate.[2]

For brevity, the S&P 500 stock market index will be called "equities" and US short-term bonds "bills". Let us take the example of an investor whose initial wealth is \$1 and plans to fund a project worth \$2. Figure 1 indicates how many years are necessary to double wealth invested either in equities or bills. Unsurprisingly, time intervals shorten during equity booms, as in the 1920's or at the end of the 2000's, and lengthen during financial downturns, as in the first years of the 1930's or at the beginning of the 2000's. The figure shows wide time variations across initial investment dates. It took only 2 years to double wealth invested in equities in 1927. Two years later, the same operation required 23 years.

Time intervals for bills were quite stable around 16 years until 1919, then

[1] Available in Shiller's webpage http://www.econ.yale.edu/∼shiller/data.htm.

[2] Treasury bills would have provided a better proxy for the riskless return than commercial papers and certificate of deposits but were not available before 1920.

experienced a rapidly increase to 33 years in 1922, a short stabilization until 1933 and a slow-moving gradual decline to a low 6 years in 1979. Time intervals rose again until 1994, which is the last investment date for which doubling wealth were possible before the series' end in 2019. Bills' yields seem especially volatile in periods of low returns during which even small variations entail large swings in duration. This was the case between 1920 and 1940 during which average yearly return was low (1.7%).

I next quantify how the two assets compare in terms of mean duration and time horizon risk. Table 1 presents summary statistics on time lengths required to accumulate various final amounts of wealth starting with $1. Results are summarized in five empirical findings.

**Finding 1.** Mean duration is around 60% longer with bills.

The "bill delay premium" (see Table 1) is the average additional delay expressed in percentage for investors to meet their wealth target when they invest in bills rather than in equities. The gap is significant and may deter impatient investors from investing in low-yield short-term bonds. Finding 1 is a consequence of the historical equity risk premium. In the data, equity real returns are on average 3.7% higher than bills' returns.

**Finding 2.** Time horizon risk increases with target wealth.

The higher the wealth target relative to initial investment the more uncertain the investment horizon, both with equities and bills. Finding 2 parallels the well-known fact that final wealth risk increases with horizon.

**Finding 3.** Time horizon risk is higher for bills than equities.

Bills perform worse than equities with respect to both mean duration and standard deviation, whatever target wealth. Finding 3 does not accord well with mean variance portfolio theory according to which investors accept to buy high risk assets in exchange of better expected returns. The classical mean variance trade-off faced by target date investors does not seem to be a trade-off for target wealth investors (more on this issue next section).

Bills' underperformance is visually confirmed in Figure 1. Bills' risk is of different nature however. They display extremely low frequency variations with only one peak over the whole period, whereas equity's risk is mainly driven by the business cycle.

**Finding 4.** Time horizon risk is positively skewed.

Duration skewness matters for time prudent investors. Ebert (2021) shows in experiment that a large majority of subjects dislike positive skewness, a situation which appears when the duration distribution has a long and fat right tail. Duration skewness is positive and decreases with target wealth both for equities and bills. Since skewness is higher for bills than for equities for all target wealths, bills are riskier on this dimension as well, which strengthens Finding 3.

**Finding 5.** Duration correlations between equity and bills are negative.

Whatever target wealth levels, longer than average investment horizon for one asset tends to be associated with shorter than average horizon for the other. Correlations increase with target wealth, suggesting significant diversification gains in mitigating duration risks, especially for high target wealth investors. Negative correlations of durations mirror negative correlations between equities' and bills' yearly returns ($-0.129$ over the whole period).

Statistics on durations do not map exactly into statistics on returns. Mean durations, duration variance and duration skewness depend on return statistics but compounded over variable investment periods. Return risks are for instance traditionally compared by computing return variance over a fixed time frame.

Since bills entail on average longer investment periods, duration variances are computed over longer sequences of returns than equities. This partially explains why bills are riskier than equities.

## 2 Mean variance approach

The textbook mean variance portfolio choice model is a useful benchmark to start with. It is analytically simple, delivers deep intuitions about risk-return trade-offs or diversification, and allows insightful graphical interpretations. The original model is framed in terms of final wealth but can be recast in terms of durations where "mean" refers to mean duration, "variance" to duration variance and return risk diversification to "horizon risk diversification".

Let us assume that investors care only about expected duration $E(t)$ and duration standard deviation $\sigma(t)$ with $t$ the time length necessary to attain a given target wealth. Investors' utility function is $U(E(t), \sigma(t))$ with $U_1 > 0$ (they prefer short durations) and $U_2 < 0$ (they dislike duration spreads around mean).

There are two assets in the economy: equities, which period $s$ return is $R_s^r$ and fixed income assets, which return is $R_s^f$. Asset returns are stationary and independently distributed over time.

Given an equity share $\alpha$, portfolio's cumulative return factor between dates 0 and $t$ is:

$$R_{0\to t}(\alpha) = \left(\alpha R_s^r + (1-\alpha) R_s^f\right)^t$$

Investors make a one-time contribution $w = 1$ at date 0 and wait until their wealth doubles. Let $\pi(t, \alpha)$ be the probability, conditional on equity share $\alpha$, that accumulated wealth reaches \$2 at date $t$ for the first time:

$$\pi(t,\alpha) = \text{Prob}\Big[\Big(R_{0\to 1}(\alpha) < 2\Big) \bigcap \Big(R_{0\to 2}(\alpha) < 2\Big) \bigcap ... \bigcap \Big(R_{0\to t}(\alpha) \geq 2\Big)\Big] \quad (1)$$

The first and second moments of the date at which wealth doubles are:

$$E(t,\alpha) = \sum_{t=1}^{\tau} \pi(t,\alpha)t$$

and

$$V(t,\alpha) = \sum_{t=1}^{\tau} \pi(t,\alpha)(t - E(t,\alpha))^2$$

with $\tau$ the maximum date at which the wealth doubles. Investors choose the equity share $\alpha$ which maximizes their mean-variance utility:

$$\max_{\alpha} \; U\Big(E(t,\alpha), \sqrt{V(t,\alpha)}\Big)$$

This simplified model can be calibrated on real data. Based on the assumption that the means and variance-covariance matrix of annual real returns for bills and stocks from 1871 to 2019 represent the distribution of future returns, Figure 2 describes how duration first and second moments change when the mix between the two assets varies. Duration is defined as the time length required for wealth to double. The curve connects feasible portfolios in the mean/standard deviation space when the equity share varies from 0 to 100% by steps of 5%. For every equity share, rolling window statistics on historical durations are computed over the period 1871-2019.

The curve is ellipse-shaped, as in classical Markowitz portfolio theory. The bold part of the portfolio curve is equivalent to Markowitz's efficient frontier in the duration domain. It excludes the gray upper arc rejected by timing risk averse investors. The minimum variance portfolio is composed for two thirds of bills (65%) and one third of equities. As already noted, the 100% equity portfolio dominates the 100% bill portfolio both in terms of expected duration and standard deviation. This does not mean however that the optimal portfolio should not include bills. Table 1 shows strong negative correlations between equities' and bills' durations, indicating that investors may achieve substantial horizon risk diversification. Starting from the all-equity portfolio, for which duration standard deviation is 6.1, reducing equity share decreases horizon risk by a significant margin, down to 3.7 if investors buy the minimum variance portfolio.

Efficient portfolios involve a substantial share of equities with a minimum of 35%. The figure indicates as an illustration a possible optimal portfolio P, composed of 55% of equities and for which investors' indifference curve (IC) is tangent to the portfolio curve. The efficient frontier looks relatively flat beyond P, which suggests that only investors sufficiently tolerant to timing risk would be willing to invest a larger fraction of their wealth in equities.

Figure 3 plots feasible portfolios in the mean/standard deviation space for target wealths ranging from \$1.5 to \$4, starting with \$1. Mean durations and duration risks are both increasing with target wealth, but mean duration increases visually faster than duration risk. Long-term investors (with high target wealth) face higher duration risk than short-term investors but only by a modest margin.

Table 2 illustrates the point by focusing on minimum variance portfolios. Those portfolios are an interesting reference as all risk averse investors choose a higher equity share. Duration standard deviation is barely increasing with target wealth. Switching from a target wealth of \$1.5 to \$4 corresponds to a 166% money goal increase and a 188% increase of mean duration (from 8.5 to 24.1 years), but a modest 25% increase in duration standard deviation (from 3.2 to 4.0). In accordance with this pattern, minimum variance equity share increases with target wealth, starting from 32% for a target wealth of \$1.5, to

47% for a target wealth of \$4. Those results suggest that the optimal equity share is likely to decrease with distance to target wealth, a property confirmed next section.

Overall, the mean variance approach shows strong horizon risk diversification benefits, points to significant shares of equities and suggests large time diversification gains associated with higher wealth target and longer investment time horizon.

# 3 Portfolio choice with Generalized Preferences

The mean-variance portfolio choice model catches basic intuitions about efficient portfolios in presence of duration risk. Like the original Markowitz model however, it is not fully dynamic (investors cannot vary their portfolio share over time) and assumes ad hoc preferences. The next sections investigate an intertemporal model of portfolio choice with better founded preferences.

## 3.1 Generalized Expected Discounted Utility

Let us define the set of dates $T = \{0, 1, ..., \tau\}$. Intertemporal utility is defined over consumption streams $C = (c(0), c(1), ..., c(\tau))$:

$$U(C) = \sum_{T} D(t)u(c(t)) \tag{2}$$

with $u$ a strictly increasing and continuous function from $[\underline{c}, \bar{c}] \subset \mathbb{R}_+$ to $\mathbb{R}_+$ and $D$ a strictly decreasing function from $T$ to $[0, 1]$.

Consumption risks are represented, without loss of generality, as risks over consumption streams $C$, which in turn makes intertemporal utility uncertain.

Generalized Expected Discounted Utility (GEDU) preferences[3] assume that individuals maximize $E\,\phi(U(C))$, where $\phi$ is a strictly increasing function from $U(C)$ to $\mathbb{R}$. An affine $\phi$ function means that individuals are risk neutral with regard to intertemporal utility uncertainty, which corresponds to the Expected Discounted Utility (EDU) model.

When $\phi$ is not affine, the GEDU representation is useful to disentangle attitude towards intertemporal consumption smoothing, captured by the curvature of $u$, from attitude towards date $t$ consumption risk, captured by the curvature of $\phi \circ (D(t)u)$. The shape of the discount function $D$, which governs impatience, also affects risk preferences when the consumption date is the unique source of uncertainty through the curvature of $\phi \circ (Du(w))$, with $w$ a fixed consumption level. The fact that $\phi$ influences both aversion to static consumption risk and aversion to consumption timing risk is in accordance with experimental evidence (see Dejarnette et al., 2020).

GEDU preferences are particularly fitted to study how individual value consumption date risk when the consumption date is random but not the consumption level. In this case, preferences simplifies to $\mathbb{E}\,\phi(D(t)u)$ with $u$ a fixed utility level. In the EDU model (when $\phi$ is affine), $D(t)$ governs both impatience and attitude towards timing risk. There is no compelling reason why preferences in two distinct domains should be determined by the same functional. In fact, EDU implies a counter-intuitive preference for random timing under the weak assumptions that $D(t)$ is decreasing and convex. Dejarnette et al. (2020) find in experiment that the vast majority of subjects are averse toward timing risk, i.e. prefer a sure delivery date to a mean preserving random date, which contradicts the EDU model.

[3] The representation is an application of the multi-attribute function of Kihlstrom and Mirman (1974) to the context of time. The term GEDU has been coined by Dejarnette et al. (2020). See also Dillenberger, Gottlieb and Ortoleva (2019).

## 3.2 Investment strategy

There are two assets: equities and fixed income assets. Given a temporal sequence of equity shares $(\alpha_0, \alpha_1, ..., \alpha_{t-1})$, portfolio's cumulative return factor between dates 0 and $t$ is:

$$R_{0\to t}(\alpha_0, \alpha_1, ..., \alpha_{t-1}) = \prod_{s=0}^{t-1} \Big(\alpha_s R_s^r + (1-\alpha_s) R_s^f\Big) \tag{3}$$

The model allows investors to dynamically rebalance their portfolio. They choose $\alpha_s$ at the beginning of every investment period and anticipate that their portfolio will be optimally rebalanced in future dates. I abstract from transactions costs and forbid borrowing or short sales. I make the simplifying assumption that asset returns are independently distributed over time. Bills' and equity's real returns are actually serially correlated in the data[4], which could make possible to predict future return based on past returns, and to condition equity share on this information. How does return predictability affect optimal portfolio choice would deserve a separate extension which is left for future research. The assumption of serial independence does not mean that the set of investment opportunities is the same every period however. As cumulated wealth gets closer to target wealth, investment opportunities may vary, which may affect the optimal equity share.[5]

Investors make a one-time contribution $w < 1$ at date 0 and wait until their wealth attains \$1. They first choose the equity share $\alpha_0$ which maximizes their expected discounted utility of wealth:

$$\max_{\alpha_0} \sum_{t=0}^{\tau} \pi_t(\alpha_0, \alpha_1^*, ..., \alpha_{t-1}^*)\phi\Big(D(t)u(1)\Big) \tag{4}$$

[4]Many studies find that expected stock returns are countercyclical. See e.g. Fama and French (1989), Ferson and Harvey (1991), Harrison and Zhang (1999), or Golez and Koudijs (2018).

[5]A similar reasoning holds for target date strategies in which the exit date is fixed and final wealth is uncertain. As the final date approaches, the distribution of cumulated final return changes, and so does optimal equity share. The equity share remains constant only for constant relative risk aversion investors with a fixed interest rate (Samuelson, 1969, Merton, 1969).

where $\alpha_1^*, ..., \alpha_{t-1}^*$ are optimal equity shares at dates 1 until $t-1$, which are the argmax of the same maximization problem considered at later time points. $\pi_t(.)$ is the probability that accumulated wealth reaches \$1 at date $t$ for the first time:

$$\pi_t(\alpha_0, \alpha_1, ..., \alpha_{t-1}) = \text{Prob}\Big[\Big(R_{0\to 1}(\alpha_0)w < 1\Big) \bigcap \Big(R_{0\to 2}(\alpha_0, \alpha_1)w < 1\Big) \bigcap ...$$
$$\bigcap \Big(R_{0\to t-1}(\alpha_0, \alpha_1, ..., \alpha_{t-2})w < 1\Big) \bigcap \Big(R_{0\to t}(\alpha_0, \alpha_1, ..., \alpha_{t-1})w \geq 1\Big)\Big] \quad (5)$$

Conditional on target wealth still to be reached, date $s > 1$ optimal equity shares are chosen the same way $s$ periods forward. As the only source of uncertainty is the consumption date, portfolio choice does not depend on static risk aversion captured by period utility $u$. Moreover, under the assumption that financial returns are serially uncorrelated, it does not depend on past returns either. A distance effect, expressed as a function of the remaining gap between current and target wealth, may nevertheless exist. Mean-risk trade-off for consumption date depends on how far current wealth is from its target. To capture the distance effect, the optimal target wealth portfolio choice is defined as a timeless function $\alpha(x)$ from $[w_0, 1)$ to $[0, 1]$ which gives for each current wealth $x$ the optimal portfolio's equity share.

The optimal equity share $\alpha(x)$, conditional on current wealth $x$, can be derived from the timeless optimal value function $V(x)$, which is the stationary expected discount appended to the utility of eventually consuming 1:

$$V(x) = \max_{\alpha} \int_{\underline{R}}^{\bar{R}} V(R(\alpha)x) f(R(\alpha)) dR(\alpha) \phi(D(t)u(1)) \quad (6)$$

where $f(R(\alpha))$ is the density function of investor's yearly portfolio return conditional on equity share $\alpha$ and $\underline{R}$ and $\bar{R}$ denote minimum and maximum portfolio returns. The simulation method, presented in Appendix, exploits the stationarity of the function $V$.

Note that as the value function only depends on wealth $x$, $V(x)$ is defined recursively. The present decision maker uses the same value function as the one used at future dates. But since the generalized discount function is not exponential, the present decision maker wrongly assumes that the discounted decision problem will be preserved one period later (Strotz, 1955). This leads to time-inconsistent portfolio strategies. Time-consistent equilibria have also be analyzed in the literature. For example, Luttmer and Mariotti (2003) considers portfolio selection problem with general discounters within the intra-personal equilibrium framework. Basak and Chabakauri (2010) analyzes the time-consistent mean-variance portfolio problem and derive its time-consistent solution using dynamic programming. Our assumption of time-inconsistency is in line with a large share of the literature in financial and saving decisions which points to various gaps between what people believe they will decide and how they actually behave (Barberis and Thaler, 2003).

The problem may also encounter issues regarding the existence and uniqueness of equilibrium in the presence of non-exponential discounting. For instance, Ekeland et al. (2012) found that a portfolio selection problem may have multiple solutions. Tan et al. (2021) demonstrated that desirable solutions for an investment problem may not exist if the discount function is non-exponential. While the existence and uniqueness are not proven in the current setup, simulations indicate a rapid convergence towards a unique solution.

Last, Program (4) ignores the possibility that investors may outreach their target and eventually consume more than \$1. For now, I will assume that the investment unit period is sufficiently short so that the assumption of constant utility may be acceptable. This issue is more rigorously addressed in the simulation method presented in Appendix by introducing fractional investment dates.

# 4 Simulated Portfolio choice

This section presents how the portfolio choice model is simulated based on the dataset used in Sections 1 and 2.

## 4.1 Functionals

For riskless intertemporal trade-offs, investors are supposed to be exponential discounter: $D(t) = \beta^t$. Other discount functions have been used in the literature. O'Donoghue and Rabin (2015) shows that functions which display present bias, like the hyperbolic discounting function are supported by evidence from various domains. Chabris, Laibson and Schuldt (2016) discusses various factors, such as uncertainty and impulsivity, that may lead to deviations from exponential discounting. Quah and Strulovici (2013) proposes a "logarithm order" to rank the impatience degrees represented by the discount functions. Prelec (1989), based on the notion of deceasing impatience, uses another parameter to rank discount functions. Ebert et al. (2020) study the discount functions using stochastic dominance and link their result to the impatience order introduced by Quah and Strulovici (2013) and Prelec (1989). Although the exponential discount function only explains a limited set of decision patterns, priority was given to tractability and benchmarking to common models of decision making.

For intertemporal utility trade-offs, $\phi$ takes a constant relative risk aversion form with risk aversion parameter $\gamma > 0$:

$$\phi(U) = \frac{U^{1-\gamma}}{1-\gamma}$$

for $\gamma \neq 1$, and $\phi(U) = ln(U)$ for $\gamma = 1$. The target wealth strategy (4) writes :

$$\max_{\alpha_0} \sum_{t=0}^{\tau} \pi_t(\alpha_0, \alpha_1^*, ..., \alpha_{t-1}^*) \frac{\beta^{t(1-\gamma)}}{1-\gamma} \tag{7}$$

with normalization $u(1) = 1$ and $\pi_t(.)$ defined in (5). In the case $\gamma = 1$, investors are risk neutral with respect to consumption timing, as can be seen from their maximization program:

$$\max_{\alpha_0} \; ln(\beta) \sum_{t=0}^{\tau} \pi_t(\alpha_0, \alpha_1^*, ..., \alpha_{t-1}^*)\, t$$

Impatient investors ($\beta < 1$ or $ln(\beta) < 0$) minimize expected duration regardless of duration's dispersion. If $\phi$ is more concave than log ($\gamma < 1$)[6], investors are timing risk averse (Dejarnette et al., 2020). They seek mean preserving date randomization if $\phi$ is less concave than log ($\gamma > 1$).

## 4.2 Calibration

Two behavioral parameters are assigned numerical values. The discount factor is set to 0.953 to match the average real annualized riskless rate of 4.88% computed from Shiller's dataset over the period 1871-2019.

The relative risk aversion coefficient (RRAC) $\gamma$ drives aversion toward intertemporal utility risk. Although values for the RRAC in the context of static consumption risks are extensively documented, this is not the case in the domain of intertemporal utility risk. In accordance with experimental results of Dejarnette et al. (2020), I restrict the analysis to the case of timing risk aversion ($\gamma \geq 1$) and select a broad range of values between 1 and 31.

## 4.3 Results

Figure 4 shows optimal equity shares given relative distance between current and target wealth. The stronger the risk aversion parameter $\gamma$, the smaller the equity

[6]The function $\phi$ is more concave/convex than log if $\phi = f \circ ln$ for some concave/convex $f$.

share. Risk neutral investors characterized by $\gamma = 1$ stay close to 100% for a large wealth interval. In contrast, investors with strong aversion to intertemporal utility risk ($\gamma \geq 6$) are heavily invested in bills. Investors with intermediate level of risk aversion ($\gamma$ between 2 and 5) choose a more balanced portfolio.

In addition, the equity share substantially varies with relative distance to target wealth, which supports the existence of a distance effect, reminiscent of the time horizon effects in target date strategies (Barberis, 2000). Investors reduce their equity share when their accumulated wealth approaches the target, but in a non linear manner. The equity share starts from 100% for moderately risk averse investors ($\gamma$ between 1 and 3) far from their target. Equity shares plateau over a wide range of wealths for all risk aversion levels, before converging to zero close to the target. The convergence to zero equity is explained by the fact that investors bear the downside cost but do not benefit from the upside beyond the wealth target.[7]

The influence of the discount factor on portfolio choice is illustrated in Figure 5 where $\gamma$ is set to 11. The time discounting parameter $\beta$ takes several values ranging from 0.900 to 0.999. We observe a similar pattern of risk profiles with a broad plateau before convergence to zero equity. The less impatient, the more willing investors are to invest in risky assets. Investors close to time neutrality ($\beta = 0.999$) invest nearly 100% of their wealth in equities before gradually disinvesting. More impatient investors invest modest amounts in equities. To understand why, recall that impatient investors would like to consume sooner than later. This fosters the demand for equities and entails shorter average delays than bills. However, impatience also makes investors more risk averse in presence of consumption date uncertainty as long investment periods are neces-

[7]Investing in equity near the target is much like being the owner of a debt contract in an equity fund.

sary to recoup financial losses. The second effect dominates the first one and makes impatient investors reluctant to invest in equities.

## 5 Conclusion

Many households pursue a personal goal when they invest, like buying a house or a car, and doing so, target a final wealth. This paper studies the case in which investors fully commit to their money goal and face an uncertain investment time horizon. Several interesting results emerge based on US long-term return data. First, fixed income assets do not appear to be less risky than equities in the duration domain. Since fixed income assets have low returns, small variations may trigger large and long-lasting swings in durations. Second, a CAPM-like model of portfolio choice graphically shows how duration risk can be substantially reduced by adding a significant share of equities in the portfolio. This illustrates another benefit of portfolio diversification beyond its ability to reduce return risk. Third, investors stay away from equities out of risk aversion, but also of impatience, as more impatient investors are more sensitive to horizon risk. Fourth, the optimal risk management strategy derived from a dynamic model of portfolio allocation consists in decreasing the equity share with relative distance to target wealth. The decrease is non-linear with a broad intermediate wealth range over which the equity share remains approximately constant. Those results provide a useful guide to financial planners who counsel households committed to a money goal. From a regulatory perspective, the risk profiling questionnaire should specifically assess investor's willingness to take timing risk and their propensity to wait, besides their tolerance to financial risk.

Several extensions of this study would be worth exploring. In the CAPM-like model, it would be interesting to add additional assets, like bonds or equities sorted by book-to-market or size. A risk-free rate could also be introduced. Investigating the issues of the existence of a market portfolio and a two-fund separation theorem would constitute an important step for future research. Price multiples such as the dividend-to-price ratio predict future return (e.g. Golez and Koudijs, 2018) but could also predict future investment durations. It would be interesting in the dynamic portfolio model to condition the equity share not only on the relative distance to target wealth, but also on a well-chosen financial ratio.

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# Appendix

## Simulation method for target wealth strategy

The optimal equity share $\alpha(x)$, conditional on current wealth $x$ is derived from $V(x)$, the expected discount attached to consuming 1. $f(R(\alpha))$ is the density function of investor's yearly portfolio return conditional on equity share $\alpha$. $V(x)$ is defined recursively:

$$V(x) = \max_{\alpha} \int_{\underline{R}}^{\bar{R}} \beta^{1-\gamma} V(R(\alpha)x) f(R(\alpha)) dR(\alpha) \tag{8}$$

Minimum and maximum portfolio returns $\underline{R}$ and $\bar{R}$ are calibrated using actual minimum and maximum returns found in the historical sample. Target wealth's value is normalized to $V(1) = 1/(1-\gamma)$. Since last period compound wealth may actually be greater than target wealth, we need to adjust at the margin the consumption date. The more wealth in excess at date $t$, the earlier investors can sell and consume before $t$, but still after $t-1$ since by assumption $x < 1$. Excess wealth ratio is defined as:

$$\varepsilon(\alpha) = \frac{R(\alpha)x - 1}{R(\alpha)x - x} \in [0, 1)$$

The date advancement is assumed to be proportional to excess wealth ratio:

$$V(R(\alpha)x) = \beta^{-(1-\gamma)\varepsilon(\alpha)} V(1) \tag{9}$$

As an illustration, suppose that last period return $R(\alpha)x - x$ is \$0.036 from which excess wealth is $R(\alpha)x - 1 = \$0.009$. Excess wealth ratio is $\varepsilon(\alpha) = 0.25$. Under the proportionality assumption, consumption takes place 3 months in advance of date $t$.

Current wealth $x$ is discretized over a fine and equally spaced grid $W = [w_0, w_0 + \epsilon, ..., 1 - \epsilon, 1]$ where $\epsilon$ is the unit interval length and $w_0$ is wealth's lower bound. The value function (8) is discretized and becomes:

$$V(x) = \max_{\alpha} \sum_{y \in W} \text{Prob}\Big(R_s(\alpha)x = y\Big)\beta^{1-\gamma}V(y) \tag{10}$$

with $V(y)$ defined by (9) if wealth overshoots its target ($y = R_s(\alpha)x \geq 1$).

The maximization problem is solved downward, starting from $x = 1 - \epsilon$. For any wealth level $x$, most next period values $V(y)$ satisfy $y = R_s(\alpha)x > x$ and have been estimated in earlier steps of the downward simulation. Values $V(y)$ satisfying $y = R_s(\alpha)x > 1$ are determined by (9). In periods with negative return ($y = R_s(\alpha)x < x$), $V(y)$ is unknown in the first iteration. It is approximated by solving a simpler portfolio choice problem in which the equity share is optimized conditional on current wealth $x$ but remains constant in subsequent periods.

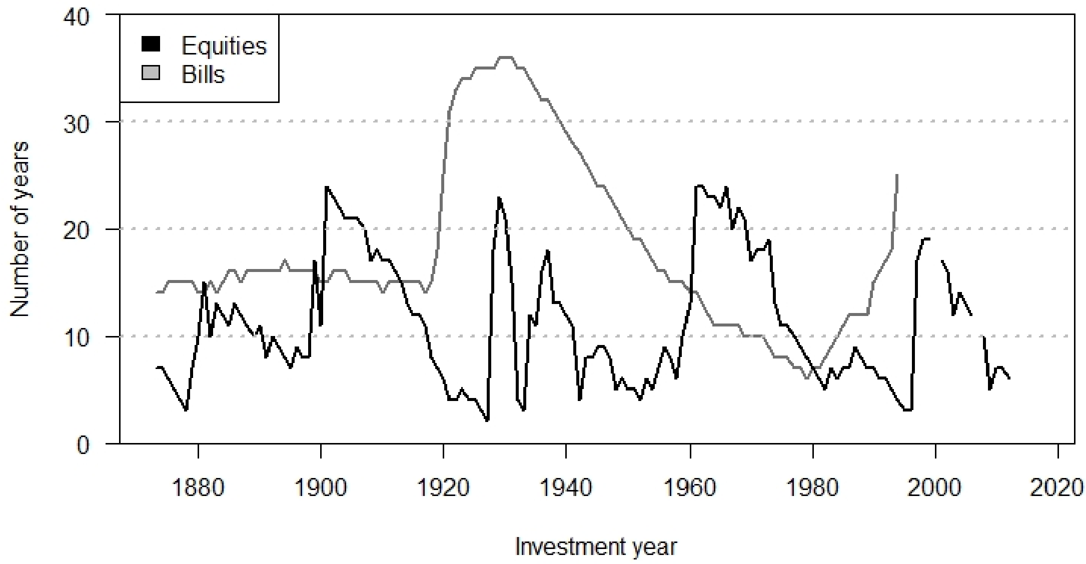
With estimated values $V(y)$, the maximization problem (10) is solved by computing the sample mean:

$$V(x) = \max_{\alpha} \frac{1}{N} \sum_{h=1}^{N} \beta^{1-\gamma} V\Big(R_h(\alpha)x\Big) \tag{11}$$

with $R_h(\alpha) = \alpha R_h^r + (1-\alpha)R_h^f$ year $h$ portfolio return. The optimal equity share $\alpha(x)$ maximizes $V(x)$ over an equally spaced grid $A = [0, \varepsilon, 2\varepsilon, ..., 1]$ for every $x$ over $W$. Because $V(y)$ is only roughly estimated in the fist round when $y < x$, several rounds are needed until $V(y)$ converges to stable values.

In simulations, the space of portfolio shares contains 1001 values with $\varepsilon = 0.01$. The space of current wealth $W$ contains 800 values with $w_0 = 0.2$ and $\epsilon = 0.001$. When $x$ is close to the lower bound $w_0$, next period wealth $y = R_h(\alpha)x$ can sometimes be lower than $w_0$. In those cases $V(y)$ is artificially set to $V(w_0)$. Because downside risk is muted, this convention biases $\alpha(x)$ upward for $x$ in the neighborhood of $w_0$. This is why all equity shares computed for $x \in [0.2, 0.25)$ are discarded in the results. Extensive testing shows that the bias is significant in the close neighborhood of 0.2 but vanishes with a good safety margin above 0.25.

Figure 1: *Number of years to double wealth, US 1871-2019*



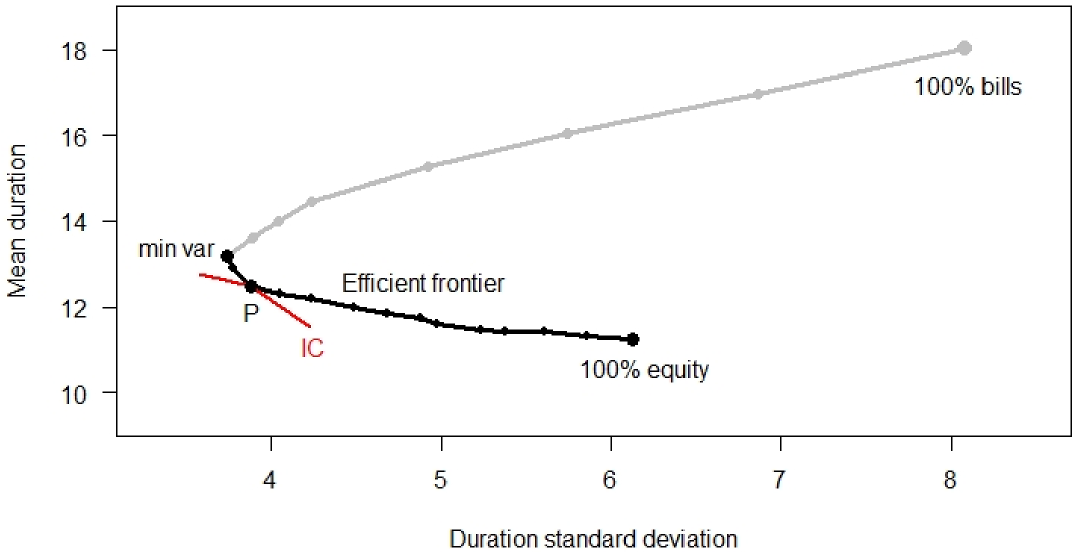

Notes. The graphic indicates how many years are necessary to double initial wealth by investing either in equities or bills for starting years beginning in 1871 and ending in 2015. Some points are missing after 2000 as wealth does not double before the last year in the dataset.

Table 1: Summary statistics for equity and bills, US 1871-2019

| Target wealth for $1 invested | 1.25 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|---|
| | | | | Equities | | | |
| Mean duration | 4.5 | 7.3 | 11.2 | 13.8 | 16.3 | 18.6 | 20.4 |
| Duration standard deviation | 4.0 | 5.1 | 6.0 | 6.3 | 6.5 | 7.0 | 6.9 |
| Duration skewness | 1.6 | 1.07 | 0.60 | 0.47 | 0.31 | 0.20 | 0.02 |
| Number of rolling windows | 147 | 142 | 128 | 126 | 125 | 124 | 124 |
| | | | | Bills | | | |
| Mean duration | 7.1 | 11.5 | 18.0 | 22.8 | 26.7 | 30.0 | 32.9 |
| Duration standard deviation | 4.7 | 6.3 | 8.1 | 9.0 | 9.8 | 10.3 | 10.9 |
| Duration skewness | 1.7 | 1.37 | 0.98 | 0.68 | 0.45 | 0.26 | 0.12 |
| Number of rolling windows | 136 | 128 | 123 | 118 | 115 | 113 | 113 |
| Bill delay premium (%) | 58 | 58 | 61 | 65 | 64 | 61 | 61 |
| Duration correlation | -0.08 | -0.15 | -0.19 | -0.29 | -0.31 | -0.36 | -0.33 |

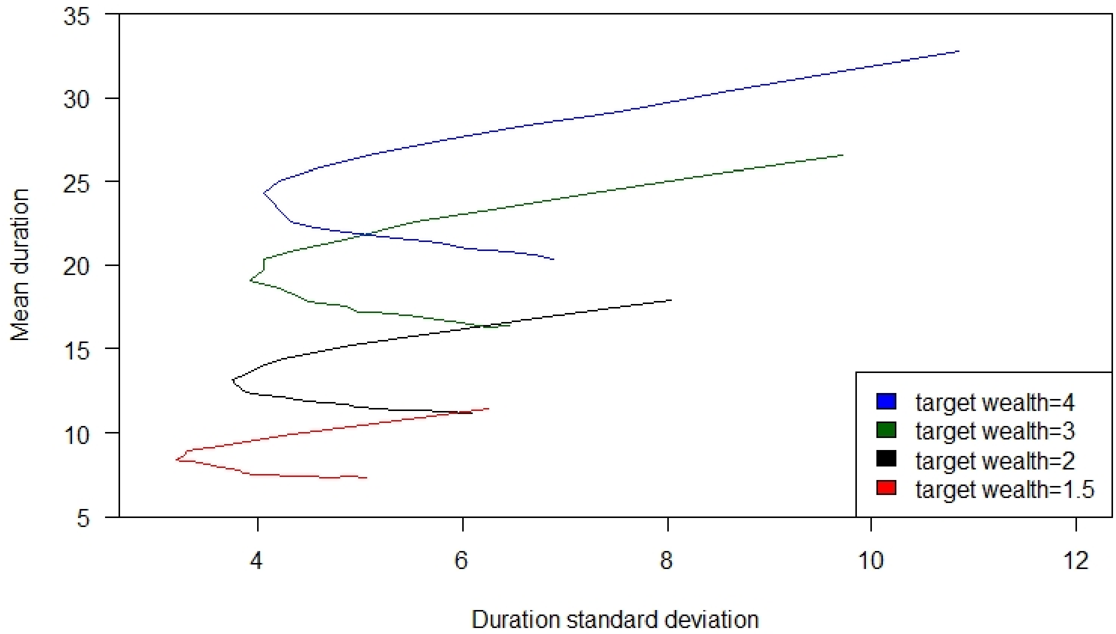
Notes. The table presents statistics about durations required to convert $1 into a final wealth ranging from $1.25 to $4. Starting from 1871 and moving the time window by one year, I compute as many historical durations as possible conditional on attaining target wealths. This provides me with a sample of durations which size is given by the number of rolling windows. Mean duration, Duration standard deviation and duration skewness are based on this sample. The bill delay premium is the additional delay expressed in percentage imposed by bills compared to equities. As durations vary across assets, duration correlations are computed for a common investment starting period.

Figure 2: *Duration mean variance frontier and optimal portfolio*




Notes. The graphic indicates feasible portfolios in the duration mean variance space for equity shares varying from 0 to 100% by step of 5%. Computation of durations needed to double wealth is based on US data, 1871-2019. The efficient frontier holds for investors preferring shorter durations and lower duration risk. Portfolio 'min var', the minimum variance portfolio, is composed of 55% of equity. P, a possible optimal portfolio given indifference curve IC, has 65% of equity.

Figure 3: *Duration mean variance curves for various target wealths*



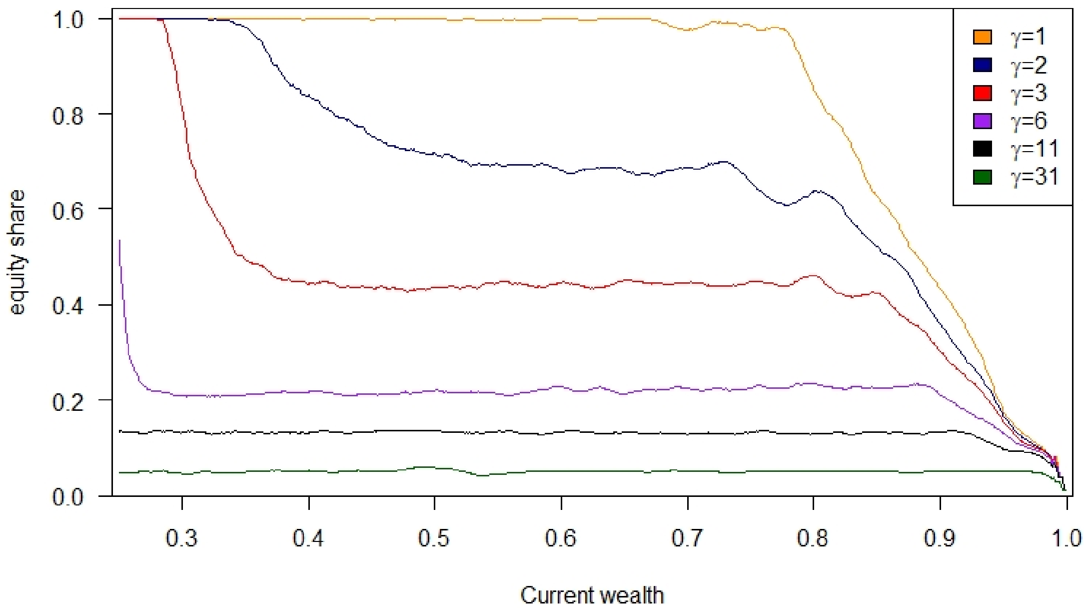

Notes. The graphic indicates feasible portfolios in the duration mean variance space for equity shares varying from 0 to 100% by step of 5% and various target wealths, starting with \$1. Computation of durations is based on US data, 1871-2019.

Table 2: Summary statistics for minimum variance portfolios, US 1871-2019

| Target wealth for \$1 invested | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|---|---|---|---|---|---|---|
| Mean duration | 8.5 | 12.9 | 16.8 | 19.3 | 21.9 | 24.1 |
| Duration standard deviation | 3.2 | 3.7 | 3.9 | 3.9 | 4.1 | 4.0 |
| Share of equities (%) | 32 | 38 | 36 | 44 | 44 | 47 |

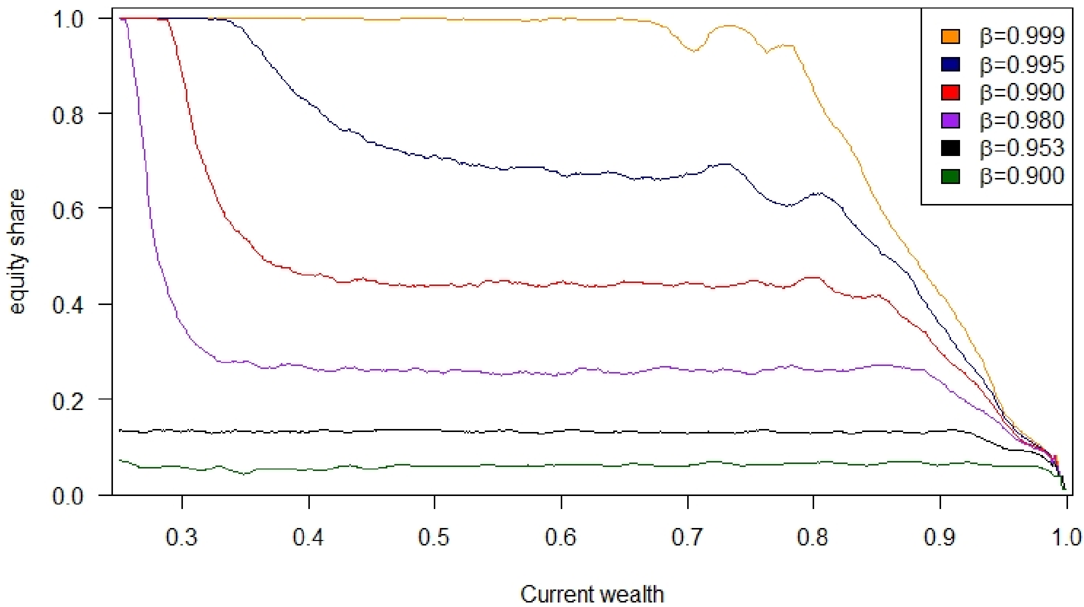
Notes. The table presents statistics and equity shares of minimum variance portfolios for target wealths ranging from \$1.5 to \$4, starting with \$1.

Figure 4: *Optimal equity share as a function of distance to target wealth and risk aversion*




Notes. The graphic indicates optimal equity shares in function of the relative gap between current and target wealth. Wealth in horizontal axis goes from 25% to 100% of target wealth. The discount factor is $\beta = 0.953$. The relative risk aversion coefficient $\gamma$ takes values ranging from 1 to 31. Simulated shares are affected by small high-frequency noise, which is smoothed out by plotting a centered moving average over a narrow rolling wealth interval of 0.02.

Figure 5: *Optimal equity share as a function of distance to target wealth and time discounting*




Notes. The graph indicates optimal equity shares in function of the relative gap between current and target wealth. Wealth in horizontal axis goes from 25% to 100% of target wealth. The risk aversion coefficient is $\gamma = 11$. The annual time discounting coefficient $\beta$ takes values ranging from 0.900 to 0.999. Simulated shares are affected by small high-frequency noise, which is smoothed out by plotting a centered moving average over a narrow rolling wealth interval of 0.02.